Prevalence of marginally unstable periodic orbits in chaotic billiards

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The dynamics of chaotic billiards is significantly influenced by coexisting regions of regular motion. Here we investigate the prevalence of a different fundamental structure, which is formed by marginally unstable periodic orbits and stands apart from the regular regions. We show that these structures both *exist* and *strongly influence* the dynamics of locally perturbed billiards, which include a large class of widely studied systems. We demonstrate the impact of these structures in the quantum regime using microwave experiments in annular billiards.

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I. INTRODUCTION

Chaotic billiards are fundamental paradigms in statistical physics and nonlinear dynamics. By connecting dynamics with geometry, billiards serve as models to address numerous questions ranging from the foundations of the ergodic hypothesis [1, 2] and the description of shell effects [3] to the design of microcavity lasers [4] and microwave resonators [5, 6], among other applications [6].

A salient feature of billiard systems is that simple geometries, such as those in Fig. 1, suffice to give rise to a rich variety of dynamical behavior observed in typical Hamiltonian systems. But as previously observed for specific chaotic billiards, simple geometries may also lead to the existence of the so-called *bouncing-ball orbits*: one-parameter families of periodic orbits exhibiting perpendicular motion between parallel walls. Theoretical and experimental work on the Sinai [Fig. 1(c)] and Bunimovich stadium [Fig. 1(d)] billiards have shown that such orbits have a major influence on transport properties, decay of correlations, and spectral properties [6, 7, 8, 9]. This is so because, contrary to the other orbits embedded in the chaotic component of the phase space, bouncing-ball orbits are only marginally unstable (i.e., perturbations grow only linearly in time). In general, marginally unstable periodic orbits (MUPOs) can be regarded as a source of regular behavior that masks strong chaotic properties. However, MUPOs are not structurally stable and may be destroyed by small changes in the parameters of the system. Therefore, MUPOs are considered to be non-generic and it has long been assumed that they could exist only for very special systems, like billiards with parallel walls.

Contrary to this expectation, in this paper we show that MUPOs are prevalent in a large class of billiard systems. The starting point of our analysis is the observation that many of the most widely studied chaotic billiards consist of *local* perturbations of an integrable billiard. For concrete examples, consider the chaotic billiards shown in the right part of Fig. 1. All these billiards can be obtained by re-defining the dynamics in the gray region of the integrable billiards in Figs. 1(a)-(c), e.g., by introducing a scatterer. It can be shown that any orbit (*i*) lying inside the chaotic component and (*ii*) not interacting with the introduced scatterers will be a MUPO. Although bouncing-ball orbits evidently satisfy these conditions in the billiards of Figs. 1(d)-(h), the existence of such orbits is far from clear in general. Here we use geometric and analytical arguments to demonstrate the widespread occurrence of MUPOs. Specifically, using circular-like billiards as model systems — such as those in Figs. 1(f)-(g) — we show that *infinitely* many families of MUPOs exist for almost all parameter choices of the system. We discuss the impact of these structures on the dynamics of chaotic orbits as well as the experimental observation of MUPOs in the quantum spectrum of microwave annular billiards.

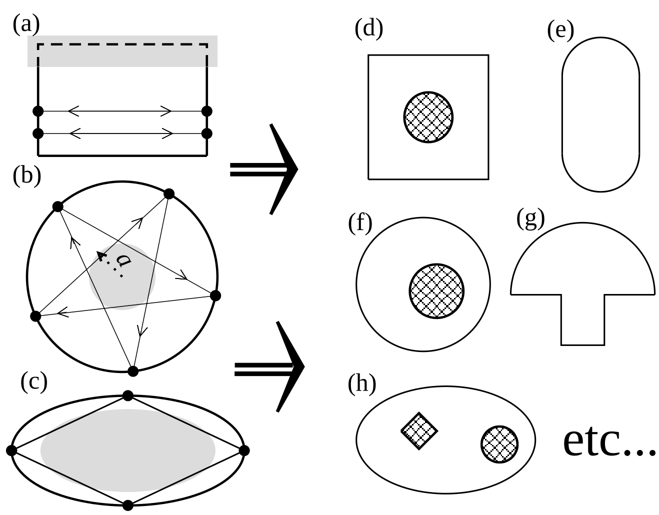

FIG. 1: Adding *local* perturbations to integrable billiards, as those shown in (a)-(c), one obtains frequently studied chaotic billiards, such as those shown in (d)-(h). The gray regions of the (a) rectangular, (b) circular, and (c) elliptical billiards are defined in such a way that chaotic motion is possible ($a < R$ in (b) is the radius of the smallest circle that circumscribes all scatterers). MUPOs are shown here to exist in billiards such as (d) Sinai [1], (e) stadium [2], (f) annular [13], (g) mushroom [14], and (h) elliptical with scatterers [15].

The *local* perturbations described above are typical for billiard systems and differ fundamentally from the *global* perturbations considered in smooth Hamiltonian systems. In the latter, the KAM theory shows that most quasi-periodic orbits of the integrable system survive the perturbation, while all periodic orbits with marginal stability disappear. Quite the opposite happens in the former case: a large set of quasi-periodic orbits disappears but there are families of periodic orbits with marginal stability that survive the perturbation by "avoiding" interaction with the localized scatterers. These orbits give rise to families of MUPOs detached from regular regions, which were previously observed in billiards with parallel walls [7], and for specific parameters of the mushroom billiard [10, 11]. Here we consider *generic* control parameters of a wide class of systems where we characterize the MUPOs both theoretically and experimentally.

The paper is organized as follows. In Sec. II we perform a detailed analysis of the existence of MUPOs in the annular billiard, a representative example of the class of billiards we are interested in. In Sec. III we show the existence of an infinite number of different families of MUPOs in annular and other circular-like billiards. Our experimental results on microwave cavities appear in Sec. IV. Finally, our conclusions are summarized in Sec. V.

II. ANNULAR BILLIARD

Annular billiards are defined by two eccentric circles, as shown in Fig. 2(a). For a fixed radius $R = 1$ of the external circle, the control parameters are the radius r and displacement $\delta < 1 - r$ of the internal circle, which serves as a scatterer. The phase space shown in Fig. 2(b) is obtained by plotting the position $\phi \in [0, 2\pi]$ of the collision of the particle with the external circumference and the sine of the angle $\theta \in [-\pi/2, \pi/2]$ with the normal direction right after the collision. In this system, periodic orbits of period q and rotation number η that collide only with the external circumference define *star polygons* of type (q, η), where the integers q and η are coprime and $\eta \leq q/2$. A star polygon of type $(5, 2)$ is shown in Fig. 1(b) and star polygons of types $(2, 1)$ and $(5, 1)$ are shown in Fig. 2. Each star polygon belongs to a family of orbits of the same type, which is parameterized by ϕ and has a fixed collision angle $\sin(\theta_{sp}) = \cos(\pi\eta/q)$.

For this system, the conditions (i) and (ii) for the existence of MUPOs mentioned in Sec. I translate into crossing the circle of radius $a = r + \delta$ without colliding with the scatterer. Under these conditions, the orbits are embedded into the chaotic sea but are only marginally unstable (both eigenvalues of the Jacobian matrix equal 1). The two orbits shown in Fig. 2 satisfy these conditions and hence are MUPOs. We use MUPOs (q, η) to denote the entire one-parameter *family* of orbits corresponding to star polygons (q, η) that satisfy conditions (i) and (ii). Note that these orbits are necessarily periodic because non-periodic orbits will either collide with the scatterer or form a regular region for $|\sin(\theta)| > a$, called whispering gallery. A collision with the scatterer happens whenever [13]

$$|\sin(\theta) - \delta \sin(\theta - \phi)| > r. \quad (1)$$

In Fig. 2b this condition is satisfied between the dashed lines. In the following we calculate the geometrical conditions for the existence of MUPOs and we demonstrate that typically an infinite number of families q, η satisfy these conditions.

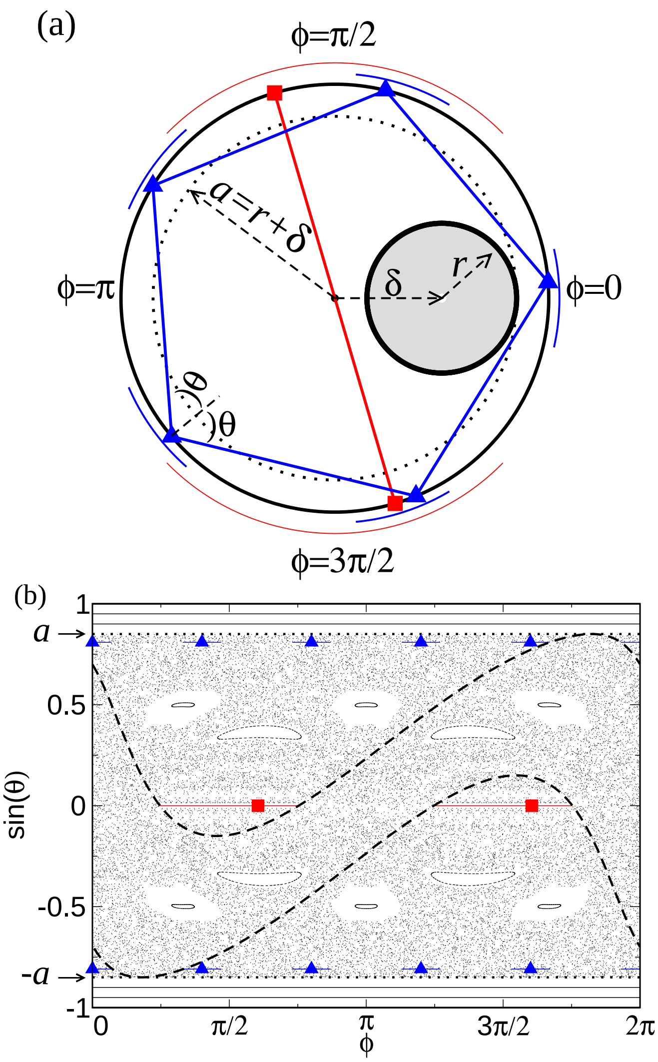

FIG. 2: (Color online) Annular billiard for parameters $r = 0.35$ and $\delta = 0.5$: (a) configuration space and (b) phase space. MUPOs correspond to the periodic orbits that cross the circle of radius a [dotted line in (a)] but that do not hit the scatterer [region between the dashed lines in (b) in which relation (1) is satisfied]. The symbols ■ and ▲ indicate, respectively, individual orbits belonging to MUPOs $(2, 1)$ and $(5, 1)$.

Consider MUPOs that encircle the scatterer from outside, like the pentagon-MUPO $(5, 1)$ in Fig. 2. Conditions for the existence of such *outer* MUPOs are obtained by noting that every star polygon (q, η) draws an inner

regular q-sided polygon, like the pentagon in Fig. 1(b). The radii (d, D) of the inscribed and circumscribed circles of this inner polygon are given by $d = \cos(\pi\eta/q)$ and $D = d/\cos(\pi/q)$. It follows that an orbit of type (q, η) is an outer MUPO (q, η) if and only if

$$\cos(\pi\frac{\eta}{q}) < \cos(\pi\lambda) \; < \; \frac{\cos(\pi\eta/q)}{\cos(\pi/q)} + r(1 - \frac{1}{\cos(\pi/q)}), \quad (2)$$

where $\cos(\pi\lambda) \equiv a = r + \delta$. A similar expression is obtained for mushroom billiards [10, 12].

Inner MUPOs, like the diameter-MUPO $(2, 1)$ in Fig. 2, exist when

$$\delta > \frac{r}{\cos(\pi(1-\eta)/q)} + \cos(\pi\frac{\eta}{q}) + \sin(\pi\frac{\eta}{q})\tan(\pi\frac{1-\eta}{q}).$$

Mixed inner-outer MUPOs may also exist for $\eta \geq 2$. Families of inner, outer, and mixed MUPOs $(5, 2)$ are illustrated in Fig. 3.

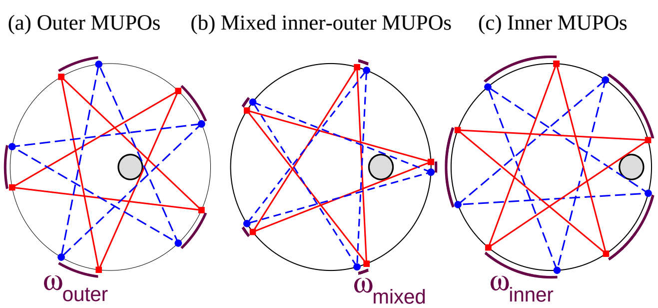

FIG. 3: (Color Online) Size w of the families of MUPOs $(5, 2)$ in the annular billiard with $r = 0.12$: (a) outer MUPOs for $\delta = 0.2$, (b) mixed inner-outer MUPO for $\delta = 0.5$, and (c) inner MUPOs for $\delta = 0.8$. All three kinds of MUPOs may coexist for a fixed δ. The size w of the families of MUPOs in Eq. (3) is given by the length of the external arcs. MUPOs $(5, 2)$ outside of these regions do not exist since they collide with the inner scatterer.

Unlike the regular regions around stable periodic orbits, MUPOs have zero Lebesgue measure in the phase space. The relevant measure is therefore the size of the families of MUPOs, given by the length w of the set of angles ϕ (normalized by 2π) for which an orbit with a given (q, η) exists. In Fig. 2, w is proportional to the length of the external arcs of circumference in (a) and to the horizontal lines in (b). For orbits inside the whispering gallery one would have $w = 1$. For a given family of MUPOs (q, η) we calculate w as

$$w = w_{outer} + w_{inner} + w_{mixed} < 1, \quad (3)$$

where $w_{outer} = 1 - q\beta^-/\pi$ and $w_{inner} = q\beta^+/\pi$ with $\cos(\beta^\pm) = [\cos(\pi\eta/q) \pm r]/\delta$. For the MUPOs (q, η) investigated in Sec. IV below, $w_{mixed} = 0$. Figure 3 shows the geometrical representation of the terms in Eq. (3).

III. INFINITE FAMILIES OF MUPOS

We now determine the number and values of the *different* families of MUPOs (q, η)'s that exist in a given annular billiard (r, δ). For inner and mixed MUPOs, only a finite number of (q, η)'s exist [16], which can be obtained by inspection. On the other hand, we show next that an infinite number of outer MUPOs (q, η) typically accumulate close to the whispering gallery. Let $\eta(q)$ denote the integer η for which $\eta/q - \lambda$ is minimal and non-negative. In the limit $q \to \infty \Rightarrow (\frac{\eta(q)}{q} - \lambda) \to 0_+$, both inequalities (2) are satisfied if

$$\frac{\eta(q)}{q} - \lambda < \frac{a\pi}{2\sqrt{1-r^2}}\frac{1}{q^2}. \quad (4)$$

Essentially the same expression is obtained for mushroom billiards [12] and the same scaling on q is expected in the case of other circular-like billiards [15].

Optimal rational approximants of $\lambda = \arccos(a)/\pi$ for a fixed q are obtained by truncating the continued fraction representation $\lambda = \frac{1}{\alpha_1 + \frac{1}{\alpha_2 + \ldots}} = [\alpha_1, \alpha_2, \ldots]$, leading to the convergent η'/q'. The irrational numbers λ^* for which there exists one integer $\alpha_{\max}$ such that $\alpha_i < \alpha_{\max}$, for all i, are called *numbers of constant type*. Numbers of constant type are difficult to approximate by rational numbers and there exist constants C_1, C_2 such that [17]

$$\frac{C_1}{q^2} < \left|\frac{\eta'}{q'} - \lambda^*\right| < \frac{C_2}{q^2}, \quad (5)$$

for all convergents η'/q'. Comparing the inequalities (4) and (5) we note the same q^{-2} dependence. Since the convergents are the *best* approximants, the lower bound in (5) is valid for all rational numbers. Therefore, provided that λ is a number of constant type, there are regions of the control parameters $[a\pi/(2\sqrt{1-r^2}) < C_1$ for annular billiards] for which there exist only a finite number of families of MUPOs. The numbers of constant type are uncountable and dense in the set of real numbers. They have zero Lebesgue measure, however, meaning that with full probability λ belongs to the complementary set of irrational numbers for which $C_2 \to 0$ in Eq. (5). Therefore, an infinite number of MUPOs exist for almost all λ and hence for almost parameters (r, δ).

The demonstration above can be used in circular-like billiards with arbitrary inner scatterers [15] to verify whether the convergents η'/q' of $\lambda = \arccos(a)/\pi$ are MUPOs (q', η') [e.g., satisfy condition (2) in the case of annular billiards or Eq. (6) of Ref. [?] in the case of mushrooms]. Typically, an infinite number of different families (q, η) can be found among the convergents. For the annular billiard illustrated in Fig. 2, for instance, all odd convergents tested are MUPOs: $(5, 1), (11, 2), (436, 77), (1342, 237), \ldots$, while the MUPO $(4, 1)$ is not a convergent.

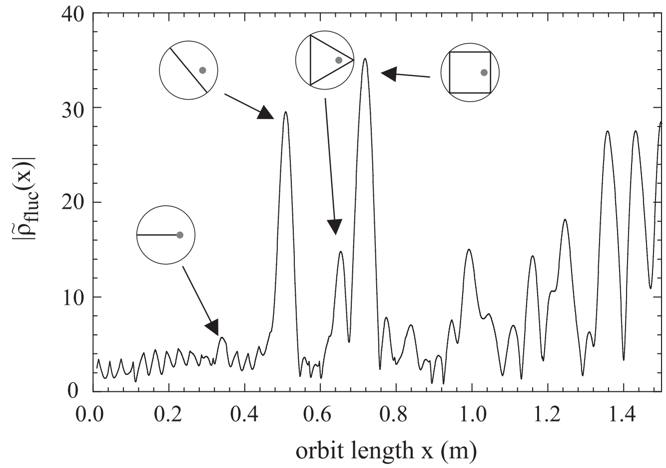

FIG. 4: Experimental length spectrum of the annular billiard for $\delta = 0.48$. Peaks associated to four periodic orbits are indicated: the shortest one is unstable, the diameter and triangle are MUPOs, and the square is inside the regular region.

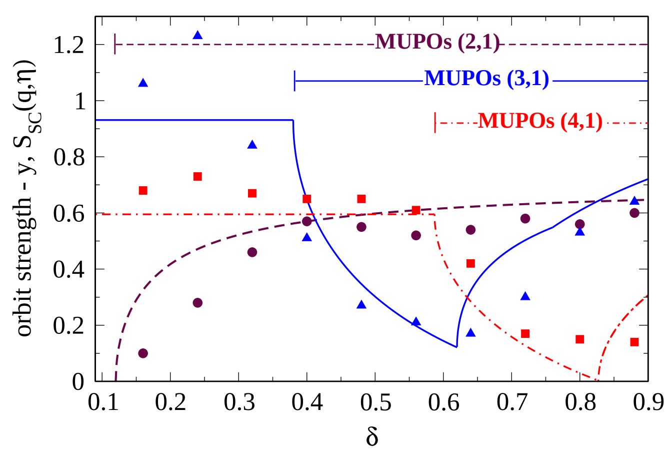

FIG. 5: (Color online) Semiclassical strengths S_{sc} (lines) compared to the experimental strengths y (symbols). S_{sc} is given by Eq. (7) and expected from periodic orbit theory, while y is extracted from the length spectra, as shown in Fig. 4. Three orbits are considered: diameter (dotted line and circles), triangle (solid line and triangles), and square (dot-dashed line and squares). The horizontal lines (top) indicate the values of δ for which the corresponding MUPOs exist.

IV. EXPERIMENTAL RESULTS

Having shown that MUPOs are abundant, we now study the impact of these structures in quantum experiments. We use the equivalence between Schrödinger's and Helmholtz's equations for flat microwave cavities [5, 6] to investigate the effect of MUPOs in quantum annular billiards. We show that MUPOs are detectable and play a prominent role among the periodic orbits.

A microwave cavity with radius 12.5 cm, height of 5 mm, and 4 coupling antennas was used in the experiments. The inner scatterer had a radius of 1.5 cm, leading to $r = 0.12$. The resonance spectra have been obtained using a vectorial network analyzer, measuring the complex amplitude ratio of the input and output microwave signal of the cavity. For each value of $\delta = 0, 0.08, \cdots, 0.88$ we measured 10 spectra up to 10 GHz with a resolution of 100 kHz. Different antennas and antenna combinations were used to find as many resonances as possible. Close lying levels (e.g., split doublets) were detected as one resonance only due to their finite width. However, since the position of those doublets can be approximately calculated, a second (*not detected*) eigenvalue could be attributed to the corresponding frequencies. We justify this procedure by using the high precision data obtained with superconducting cavities in the experiments described in Ref. [18]: there the doublets could be resolved, and we found that the length spectrum is stable under small random shifts of one doublet partner, which allows us to assume doublets to be degenerate. Finally, by comparing the number of detected levels $N(f)$ below the frequency f to the expected number $N_{\rm Weyl}$ given by Weyl's formula [3], we checked that almost all eigenvalues in the considered part of the spectrum have been found ($N \approx 150$ for each δ).

Performing a Fourier transform ($\mathcal{FT}$) of the level density $\rho(f) = \frac{dN(f)}{df}$ we have computed the length spectrum

$$|\tilde{\rho}_{fluc}(x)| = |\mathcal{FT}\{\rho(k) - \rho_{\rm Weyl}(k)\}|\,, \qquad (6)$$

where $k = 2\pi f/c$. The classical periodic orbits manifest themselves as peaks located at the corresponding orbit length. The length of periodic orbit (q,η) is given by $x_{(q,\eta)} = 2Rq\sin(\pi\eta/q)$. Particularly, for all δ's we consider the peak heights y (strengths) of the diameter ($x_{(2,1)} = 0.5$ m), triangular ($x_{(3,1)} = 0.63$ m), and square ($x_{(4,1)} = 0.71$ m) orbits. The length spectrum for $\delta = 0.48$ is shown in Fig. 4, where we indicate additionally the peak at $x = 0.34$ m related to an *unstable* periodic orbit. Notice that this peak is much smaller than the peaks associated with the MUPOs.

Using periodic orbit theory, the strength of an orbit in a quantum mechanical length spectrum is given by the amplitudes of the oscillatory terms in a semiclassical periodic orbit summation. We use the trace formula for integrable systems to obtain the orbit dependent amplitudes $\mathcal{A} = \nu\frac{\sin^{3/2}(\pi\eta/q)}{\sqrt{q}}$, where $\nu = 1$ for the diameter orbit, and $\nu = 2$ for all other orbits [3]. The expected strength in the case of the MUPOs is

$$S_{sc}(q,\eta) = w\mathcal{A}, \qquad (7)$$

where w is the measure of the entire family, given in Eq. (3) and illustrated in Fig. 3. In Fig. 5 we compare S_{sc} (lines) with the experimental strengths y (symbols, rescaled by a common factor) for different values of δ. The dependence of S_{sc} on δ is due to the factor w. Overall, the orbits strengths y approximately follow the semiclassical behavior S_{sc} for $\delta > 0.3$. The deviations can be understood qualitatively as the experiment diverges from

the semiclassical limit: (1) the finite wavelengths imply a spatial uncertainty of the order of the typical width of the peaks in the length spectrum; (2) the Fourier transform of a finite spectral range generates fluctuations in the length spectra (of the order of 10 % of the diameter peak height, as seen for $x < 0.2\,\mathrm{m}$ in Fig. 4). Nevertheless we find that quantum behavior resembles the classical behavior in the sense that the data support the use of the weighting factors w in the semiclassical strengths in Eq. (7).

V. CONCLUSIONS

We have demonstrated that MUPOs are prevalent and that they must be accounted for in billiard experiments, which is a new paradigm that advances previous conclusions drawn for specific systems [6, 8, 10]. In particular, MUPOs have not been previously observed in annular billiards, despite many theoretical [13, 21, 22] and experimental [18] studies, including detailed catalogs of periodic orbits [23]. We have shown that annular and general circular-like billiards typically have an infinite number of different families of MUPOs in the chaotic component close to the border of the whispering gallery. This should be contrasted with the case of billiards with parallel walls such as Stadium and Sinai billiards, where only a finite number of families of MUPOs exists.

The above mentioned results can be immediately extended to other chaotic billiards defined by local perturbations of integrable systems and are expected to find applications in both classical and quantum studies. Classically, general arguments on marginal instability can be used to show that the resulting stickiness of chaotic trajectories to MUPOs generates a universal power law $p(t) \sim t^{-2}$ for the survival probability of nearby particles [11]. This scaling is expected to hold for long times, while fluctuations (nonperiodic echoes) occur for short times [10, 19]. Studies in the quantum regime have shown that orbits with marginal stability are robust to small perturbations [20] and give rise to different transport phenomena [9]. Recent theoretical studies and microwave experiments on chaos assisted tunneling in the annular billiard have demonstrated a pronounced effect on the tunneling of the so called "beach region" between the whispering gallery and the chaotic region [18, 21]. Different mechanisms of dynamical tunneling are currently under investigation [24], and special attention is being devoted to mushroom billiards [25, 26]. Our results have fully characterized the dynamics in the "beach region" of annular and mushroom billiards in terms of marginal unstable orbits. The formalization of their contribution to dynamical tunneling and a comparison with the existing numerical and experimental results are interesting open questions.

Acknowledgments

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